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CONTENTS

		Pages.
C. S. Venkataraman:	On some Remarkable types of Multiplicative functions ..	1—12
B. R. Seth:	Stability of Rectilinear Plates.	13—16
V. Ganapathy Iyer:	On the Translation numbers of Integral functions ..	17—28
F. W. Levi:	The ring of Endomorphisms for which Every subgroup of an Abelian Group is Invariant.	29—31
Hansraj Gupta:	A note on the Parity of $p(n)$ ..	32—33
B. K. Sen Gupta:	On the Aberrancy Curve ..	33—34
The Narasinga Rao Prize Problem	..	34—36

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OFFICE OF THE REGISTRAR OF BOOKS
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MADRAS

ON SOME REMARKABLE TYPES OF MULTIPLICATIVE FUNCTIONS*.

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BY

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1. INTRODUCTION. In his memoir† "The Theory of Multiplicative Arithmetic Functions", Dr. R. Vaidyanathaswamy mentions certain important types of multiplicative functions of $r(>1)$ arguments, viz. (1) the ordinal, (2) the principal and (3) the modular functions.

Thus, the multiplicative function $f(M_1, .., M_r)$ is said to be an "ordinal" function if it vanishes whenever certain prescribed inequalities of the form $M_i > M_j$ hold between the arguments. M_i and M_j are called the minor and the major arguments, respectively, of the ordinal function.

$f(M_1, .., M_r)$ is called a "principal" function if it vanishes whenever any two of its arguments are unequal.

If a multiplicative function $f(M_1, .., M_r)$ has the property that its value is unaltered when M_i is increased by any multiple of M_j, it is said to be a "modular" function. M_j is called a modulus of f with respect to (or relative to) M_i.

It may be easily seen that the principal function is a particular case of the ordinal function.

Recently‡ I studied certain kinds of generalized ordinal, principal and modular functions. The object of the present paper is to show that apart from these kinds of functions, there are several other types with interesting mutual relations. The method employed here is a new

* I am indebted to Dr. R. Vaidyanathaswamy for his valuable suggestions and help in the preparation of this paper.

† *Trans. Amer. Math. Soc.* 33 (1931) 579-662. Throughout this paper, this memoir will be referred to as M.F.

‡ In a thesis submitted for the M. Sc. Degree of the University of Madras.

one, viz. that of convolution. We begin by explaining the process of convolution.*

2. CONVOLUTION. Let $f(M_1,..,M_r)$ be a multiplicative function of r (>1) arguments. Then the process of convolving M_1, M_2 in f consists in forming the function $\phi(M, M_3,..,M_r) = \Sigma f(M_1, M_2..,M_r)$, where the summation on the right side is for all values of M_1, M_2 such that $M_1 M_2 = M$. ϕ will be called a 'convolute' of f and can be easily shown to be a multiplicative function of its $(r-1)$ arguments.

Also if the generating series of f to the base p is

$$f_p(x_1,..,x_r) = \sum_{m_i=0}^{\infty} a_{m_1,m_2,..,m_r} x_1^{m_1} x_2^{m_2}..x_r^{m_r} \ (i=1,2,...,r) \text{ then}$$

that of ϕ to the same base is

$$\phi(x, x_3,..,x_r) = \sum_{m_i=0}^{\infty} \Big(\sum_{m_1+m_2=m} a_{m_1,m_2,..,m_r} \Big) x^m x_3^{m_3}..x_r^{m_r}.$$

Thus the generating series of ϕ is obtained from that of f by putting $x_1 = x_2 = x$.

The process of convolution is evidently applicable to any number S of arguments and is equivalent to the identification of the corresponding variables in the generating series. Hence in convolving several arguments, it is immaterial whether we convolve them all together or in stages.

The process of composition, viz. forming the function $f(M_1,..,M_r) = \sum_{\delta_i/M_i} f_1(M_1/\delta_1,..,M_r/\delta_\gamma) f_2(\delta_1, \delta_2,..,\delta_r)$, $(i=1,..,r)$, from $f_1(M_1,..,M_r)$ and $f_2(M_1,..,M_r)$ can be explained in terms of convolution. Thus f is obtained by convolving the r pairs $(M_i. N_i)$ in the product $f_1(M_1, M_2,..,M_r) \times f_2(N_1, N_2,..,N_r)$.

If $f(M_1,..,M_r)$ and $\phi(M_1,..,M_r)$ be two functions of the same arguments and Ω represents any convolution of

* *See* M.F. pp. 591-595.

the arguments, it can be easily shown that $\Omega[f\phi] = [\Omega f][\Omega\phi]$, where $f\phi$ denotes the composite of f and ϕ.

The composite of $f(M_1, .., M_r)$ and $E_0(M_1, .., M_i) \times E(M_{i+1}, .., M_r)$ is called the integral of f with respect to $M_{i+1}, .., M_r$, where E_0 and E are given by

$$E_0(M_1, .., M_r) = \begin{cases} 1, \text{ when } M_1 = M_2 = .. = M_r = 1 \\ 0, \text{ otherwise}; \end{cases}$$

and $E(M_1, .., M_r) = 1$ for all value of $M_1, .., M_r$.

3. We shall next prove certain general properties of ordinal functions. It is enough if we consider the ordinal function $f(M_1, .., M_r)$ which vanishes whenever an ordinal relation of the type $M_i > M_j$ holds between a single pair of arguments only, since evidently we can by repetition of the arguments deduce the properties when the ordinal relation of the above kind holds between more than one pair of arguments.

THEOREM 1. *If $f(M_1, .., M_r)$ is an ordinal function which vanishes whenever $M_i > M_j$ it also vanishes whenever M_i does not divide M_j.**

Conversely, if a multiplicative function $f(M_1, .., M_r)$ vanishes whenever M_i does not divide M_j, it is an ordinal function with M_i and M_j as the minor and the major arguments respectively.

For, $f(M_1, .., M_r) = \prod_{\lambda=1,2,..} f(p_\lambda^{a_{\lambda 1}}, .., p_\lambda^{a_{\lambda r}})$ since f is multiplicative. Now $f(p^{m_1}, .., p^{m_r})$ for any prime p vanishes whenever $p^{m_i} > p^{m_j}$ by hypothesis. Hence f vanishes when $m_i > m_j$, i.e. when a prime p occurs to a higher power in M_i than in M_j. Thus f vanishes whenever M_i does not divide M_j.

Conversely, if a multiplicative function f vanishes whenever M_i does not divide M_j, it necessarily vanishes if $M_i > M_j$, for then M_i does not divide M_j. Hence f is ordinal.

THEOREM 2. *If $f(M_1, .., M_r)$ is an ordinal function and $\phi(M_1, .., M_r)$ is any multiplicative function, then (a) their product*

* M. F. p. 656, Theorem XXXIX.

is an ordinal function, (b) their composite is an ordinal function with M_i, M_j as the minor and major arguments, if and only if, ϕ vanishes whenever $M_i > M_j$.

PROOF. (*a*) is obvious. To prove (*b*) let us first assume that ϕ is an ordinal function, vanishing whenever $M_i > M_j$. Let $F(M_1, .., M_r) = f\phi = \sum_{\delta_i/M_i} f(M_1/\delta_1, .., M_r/\delta_r) \times \phi(\delta_1, \delta_2, .., \delta_r)$ $(i = 1, 2, .., r)$.

Now if $M_i/\delta_i > M_j/\delta_j$, f vanishes by hypothesis; while if $M_i/\delta_i \leqslant M_j/\delta_j$, then $\delta_i = M_i\delta_i/M_i \geqslant M_i\delta_j/M_j > M_j\delta_j/M_j = \delta_j$, i.e. $\delta_i > \delta_j$. Hence ϕ vanishes.

Thus every term on the right side vanishes whenever $M_i > M_j$ and hence F is ordinal with M_i, M_j as the minor and the major arguments.

Next let us assume that ϕ is a multiplicative function which does not vanish for all values of $M_i > M_j$, i.e. there are values of M_i, M_j such that $M_i > M_j$ and $\neq 0$ for these pairs of values.

As before let F be the composite of f and ϕ. Since ϕ is not ordinal with respect to M_i, M_j, there exist a prime p and a set of members $m_1, m_2, .., m_r$ such that $\phi(p^{m_1}, .., p^{m_i}, p^{m_j}, .., p^{m_r}) \neq 0$ for $m_i > m_j$. Let S_j be the smallest value of m_j for which this is true. Suppose S_i to be the corresponding smallest value of m_i, among the set of numbers $m_1, .., m_i .., m_j, .., m_r$. Now $\phi(p^{m_1}, .., p^{S_i}, .., p^{S_j}, .., p^{m_r}) \neq 0$ and $S_i > S_j$.

Again let s_1 be the corresponding smallest value of m_1 in the set $m_1 .. s_i, s_j, .., m_r$. Clearly, in this manner we can fix the numbers $s_2, s_3, .., s_r$ also such that

$$\phi(p^{s_1}, .. p^{s_i}, .. p^{s_j}, .., p^{s_r}) \neq 0 \text{ and } s_i > s_j \qquad (1)$$

and $\phi(p^{\delta_1}, p^{\delta_i}, p^{\delta_j}, .., p^{\delta_r}) = 0$ if any δ is less than the corresponding s and δ_i greater than δ_j. (2)

Now

$$F(p^{s_1},..,p^{s_i},..p^{s_j},..p^{s_r}) =$$
$$\sum_{\delta_i} f\left(\frac{p^{s_1}}{p^{\delta_1}},..\frac{p^{s_i}}{p^{\delta_i}},..\frac{p^{s_j}}{p^{\delta_j}},..\frac{p^{s_r}}{p^{\delta_r}}\right)\phi(p^{\delta_1},p^{\delta_2},..,p^{\delta_r}),$$
$$\delta_i = 0, 1, .. s_i, \; i = 1, 2, .. r.$$

If $\delta_i \leqslant \delta_j$ then $p^{\delta_i} \leqslant p^{\delta_j}$ and therefore $\frac{p^{s_i}}{p^{\delta_i}} \geqslant \frac{p^{s_i}}{p^{\delta_j}} > \frac{p^{s_j}}{p^{\delta_j}}$. Hence f vanishes by hypothesis.

Let $\delta_i > \delta_j$. Then $\phi(p^{\delta_1}, p^{\delta_2},..,p^{\delta_i}, p^{\delta_j},..,p^{\delta_r})$ vanishes by (1) and (2), if $\delta_i \neq s_i$ $(i = 1, 2,..,r)$.

Hence the right side reduces to $\phi(p^{s_1}, p^{s_2},..,p^{s_i}, p^{s_j}.p^{s_r}) \neq 0$. Thus $F \neq 0$ for $s_i > s_j$.

Hence F is not ordinal with respect to M_i, M_j.

The importance of Theorem 2 (*b*) lies in proving the ordinal nature of the inverse of f. Thus we have

THEOREM 3. *The inverse of an ordinal function* $f(M_1, M_2,..,M_r)$ *which vanishes whenever* $M_i > M_j$, *is ordinal with* M_i, M_j *as minor and major arguments.*

Conversely if the inverse $f^{-1}(M_1,..,M_r)$ *of a multiplicative function* f *is ordinal with* M_i, M_j *as minor and major arguments, then* f *is necessarily an ordinal function of the same kind.*

For, $f(M_1, M_2,..,M_r).\, f^{-1}(M_1, M_2,..,M_r) = E_0(M_1, M_2,..,M_r)$ and E_0 is obviously an ordinal function vanishing whenever $M_i > M_j$.

Hence by (*b*) of Theorem 2 it follows that f^{-1} vanishes for $M_i > M_j$.

The second part is immediate from the fact that of the functions f, f^{-1}, each is the inverse of the other.

THEOREM 4. *The integral of an ordinal function* $f(M_1, M_2,..,M_r)$ *with respect to the major argument* M_j *is an ordinal function of the same kind, while that with respect to the minor argument* M_i *is a modular function with* M_j *as modulus, relative to* M_i.

The first part follows from Theorem 2 (*b*) since $E_0(M_1,..,M_i, M_{j-1}, M_{j+1},..,M_r)E(M_j)$ is an ordinal function which vanishes for $M_i > M_j$.

To prove the second part, let F be the integral of f with respect to M_i, so that $F(M_1,..,M_r)$

$$= f\,(M_1,..,M_r) \quad E_0(M_1,..,M_{i-1},\ M_{i+1},..,M_r) \quad E(M_i)$$
$$= \sum f(\delta_1,..,\delta_i,\ \delta_j,..,\delta_r)\ E_0\left(\frac{M_1}{\delta_1},..\frac{M_{i-1}}{\delta_{i-1}},..\frac{M_r}{\delta_r}\right) E\left(\frac{M_i}{\delta_i}\right)$$
$$= \sum_{\delta_i | M_i} f(M_1,\ M_2,..,\delta_i,\ M_r).$$

Now, by Theorem 1, f vanishes if δ_i does not divide M_j. Hence δ_i is a common factor of M_i, and M_j, i.e. a divisor of g_{ij} the greatest common divisor of M_i and M_j.

Thus, $F(M_1,..,M_r) = \sum\limits_{\delta_i | g_{ij}} f(M_1,..,\delta_i,\ M_r)$

$$= F(M_1,\ M_2,..,\ g_{ij},..,M_r).$$

Since $(M_i,\ M_j) = (M_i+\lambda M_j,\ M_j) = g_{ij}$, it follows that F has the modulus M_j with respect to M_i and that the value of F depends on M_i only through its g.c.d. with M_j.

We shall now prove the converse of the second part of Theorem 4, thereby establishing the fundamental mutual relation between the ordinal and the modular functions.

THEOREM 5. *If $F(M_1,..M_i,..,M_j,\ M_r)$ is a modular function with the modulus M_j relative to M_i, then its composite with $E_0(M_1,..,M_{i-1},\ M_{i+1},..,M_r)\ E^{-1}(M_i)$ is an ordinal function which vanishes whenever $M_i > M_j$.*

PROOF. Le $f(M_1,..,M_i,..,M_j,..,M_r)$

$$= F(M_1,..,\ M_i,..,M_j,\ M_r)\ E_0(M_1,..,M_r)\ E^{-1}(M_i)$$
$$= \sum_{\substack{\delta_i | M_i \\ (i=1,\,2,..,r)}} F\left(\frac{M_1}{\delta_1},..\frac{M_i}{\delta_i},..\frac{M_r}{\delta_r}\right) E_0(\delta_1,..,\delta_{i-1},\ \delta_{i+1},\ \delta_r) E^{-1}(\delta_i)$$
$$= \sum_{\delta_j | M_i} F(M_1,..,\ M_i/\delta_i\ ,..,M_r)\ \mu(\delta_i). \qquad (1)$$

[The inverse of $E(M)$, viz. $E^{-1}(M)$ can be easily seen to be the Möbius function $\mu(M)$.]

Now

$$f(M_1,\ldots,M_i,\ldots,M_r) = \prod_{(\lambda=1,2,\ldots)} f(p_\lambda^{a_{\lambda 1}},\ldots,p_\lambda^{a_{\lambda i}},\ldots,p_\lambda^{a_{\lambda r}}), \quad (2)$$

where

$$M_i = p_1^{a_{1i}} p_2^{a_{2i}} p_3^{a_{3i}} \ldots (i = 1, 2, \ldots r)$$
$$p_1 < p_2 < p_3 < \ldots$$

and for any prime p,

$$f(p^{m_1},\ldots,p^{m_i},\ldots,p^{m_j},\ldots,p^{m_r}) = \sum_{\delta_i / p^{m_i}} F(p^{m_1},\ldots\frac{p^{m_i}}{\delta_i},\ldots,p^{m_r})\mu(\delta_i)$$

$$= F(p^{m_1},\ldots p^{m_i}, p^{m_{i+1}},\ldots p^{m_r}) - F(p^{m_1},\ldots p^{m_i-1}, p^{m_{i+1}},\ldots p^{m_r}). \quad (3)$$

Since F is a modular function with M_j as modulus relative to M_i,

$$F(p^{m_1},\ldots,p^{m_i}, p^{m_{i+1}},\ldots,p^{m_j}\ldots,p^{m_r})$$
$$= F(p^{m_1},\ldots,p^{m_j}, p^{m_{i+1}}\ldots,p^{m_j}, p^{m_r})$$

whenever $m_i > m_j$. For let $m_i = m_j + k$ $(k \geqslant 1)$. Then

$$F(p^{m_1},\ldots,p^{m_i},\ldots,p^{m_r})$$
$$= F(p^{m_1},\ldots,p^{m_j+k}, p^{m_{i+1}},\ldots,p^{m_j},\ldots,p^{m_r})$$
$$= F(p^{m_1},\ldots,p^{m_j}+\lambda p^{m_j}, p^{m_{i+1}},\ldots,p^{m_j},\ldots,p^{m_r}),$$

where $\lambda = p^k - 1 (> 0)$

$$= F(p^{m_1},\ldots,p^{m_j}, p^{m_{i+1}},\ldots, p^{m_j},\ldots,p^{m_r}) \quad (4)$$

on account of the modular property.

Now if $m_i > m_j$, then $m_i - 1 \geqslant m_j$. Then from (3) and (4), it follows that $f(p^{m_1},\ldots,p^{m_i},\ldots p^{m_j}\ldots,p^{m_r}) = 0$ whenever $p^{m_i} > p^{m_j}$.

Therefore $f(M_1,\ldots,M_i,\ldots,M_j,\ldots M_r) = 0$ if M_i does not divide M_j and is thus an ordinal function with M_i, M_j as the minor and the major arguments respectively.

NOTE. In arriving at the result (4), we have incidentally shown that if $F(M_1,\ldots,M_i,\ldots,M_j,\ldots,M_r)$ is a modular function with M_j as modulus relative to M_i, then $F(M_1,\ldots, M_i\ldots, M_j,\ldots M_r) = F(M_1,\ldots,g_{ij},\ldots M_j,\ldots M_r)$ where $g_{ij} = \prod_{(\lambda=1,2,\ldots)} p_\lambda^{\min(a_{\lambda i}, a_{\lambda j})}$, the g.c.d, of M_i and M_j. This is a fundamental property of modular functions.

3.1. Remarks. The theorem proved above for ordinal functions in which an ordinal relation holds between a single pair of arguments only, can be clearly extended to cases where an ordinal relation holds between several pairs of arguments. We shall state the results analogous to Theorem 4, for the following typical kinds of ordinal functions, viz. (1) $f(M_1, M_2, .., M_r, N_1, .., N_s)$ in which an ordinal relation holds between k distinct pairs of arguments N_i, M_i, N_i's being the minor arguments, i.e. f vanishes whenever $N_i > M_i$ $(i = 1, 2, .., k)$ and $k \leqslant$ the smaller of r, s, and (2) $\psi(M_1, M_2, .., M_r, N_1, .., N_s)$ in which an ordinal relation holds between two groups of arguments N_i, M_j such that each argument of the first group (the group of N's) is a minor argument relative to each argument of the second group (the group of M_j's); in other words ψ vanishes whenever any $N_i >$ any M_j $(i = 1, 2, .., k \leqslant s$; and $j = 1, 2, .., l \leqslant r)$.

It would be convenient to refer to f and ψ, as "ordinal functions of the first kind, of type k" and "ordinal functions of the second kind, of type l, k," respectively.

The analogous theorems are :—

(a) *The integral of an ordinal function of the first kind, of type k, with respect to its k minor arguments* N_i $(i = 1, 2, .., k)$, *is a modular function with the moduli* M_i *relative to* N_i $(i = 1, 2, .., k)$ *(which may be called modular function of the first kind, of type k).*

(b) *The integral of an ordinal function of the second kind, of type l, k, with respect to the group of k minor arguments* N_i *is a modular function with each* M_j *as modulus relative to each* N_i *(we shall refer to this integral as modular function of the second kind, of type l, k).*

4. We now proceed to study the result of convolution of ordinal functions. We shall concern ourselves

with the two typical ordinal functions mentioned in the last para.

THEOREM 6. *The convolute of an ordinal function of the first kind and of type k, (viz. the function formed by convolving separately the k minor arguments and the k major arguments) is an ordinal function of the first kind of type* 1.

Let the convolute of f be

$$\psi(M, M_{k+1}, .., M_r, N, N_{k+1}, .., N_s) = \Sigma f(M_1, .., M_r, N_1, .., N_s),$$

where $$M = \prod_{i=1}^{k} M_i, \; N = \prod_{i=1}^{k} N_i.$$

Suppose $N > M$. The N_i cannot be less than M_i for $i = 1, 2, .., k$, simultaneously. Therefore there is at least one value of i (say α) $\leqslant k$ such that $N_\alpha \geqslant M_\alpha$. If $N_\alpha = M_\alpha$, then $\frac{N}{N_\alpha} > \frac{M}{M_\alpha}$ and by a similar reasoning, it can be shown that there is a $\beta \neq \alpha$ for which $N_\beta \geqslant M_\beta$ $(\beta \leqslant k)$. Again if N_β happens to be equal to M_β, we can repeat the argument so that for every set of numbers $N_1, N_2, .., N_k = N > M = M_1, M_2, .., M_k$, there is a value of i, such that $N_i > M_i$ for $i \leqslant k$.

Hence every term on the right side and therefore ψ vanishes if $N > M$. Thus ψ is an ordinal function of the first kind and of type 1.

THEOREM 7. *The convolute of the integral with respect to the k minor arguments of an ordinal function of the first kind and of type k (viz. the result of convolving separately all the minor arguments and the major arguments) is a modular function of the first kind of type* 1.

For if Ω represents the convolution of $N_1, .., N_k$ and $M_1, .., M_k$, and

$$F(M, M_{k+1}, .., M_r, N, N_{k+1}, .., N_s)$$

$$= \Omega[f(M_1, \ldots, M_r, N_1, \ldots, N_s) \times E_0(M_1, \ldots, M_r, N_{k+1}, N_s)\, E(N_1, \ldots N_k)]$$

then $F = \Omega f(M_1, \ldots, M_r, N_1, \ldots, N_s)$

$$\times \Omega E_0(M_1, \ldots, M_r, N_{k+1}, \ldots, N_s)\, E(N_1, \ldots, N_k)$$

$$= \psi(M, M_{k+1}, \ldots, M_r, N, N_{k+1}, \ldots, N_s) \times E_0(M, M_{k+1}, \ldots M_r, N_{k+1}, \ldots, N_s)\, E(N).$$

[It can be seen that $E_0(M, M_{k+1}, \ldots M_r, N_{k+1}, \ldots, N_s)\, E(N)$ is the convolute of $E_0(M_1, \ldots M_r, N_{k+1}, \ldots, N_s)\, E(N_1, \ldots, N_k)$.]

Hence by Theorem 4, it follows that F is a modular function with modulus M relative to N, i.e. F is a modular function of the first kind, of type 1.

COR. *The convolute of a modular function of the first kind, of type k (viz. the function formed by convolving separately the k modular arguments and the k non-modular arguments) is a modular function of the first kind, of type* 1.

For F is a modular function of the first kind, of type k by 3.1 (a), and by Theorem 5 any modular function of the first kind of type k is the integral of an ordinal function of the first kind of type k, with respect to the minor arguments.

THEOREM 8. *The convolute of an ordinal function of the second kind of type l, k (viz. the function formed by convolving separately the group of k minor arguments and the group of l major arguments) is an "indexed ordinal function."*

PROOF. Let $\psi(M, M_{l+1}, \ldots, M_r, N, N_{k+1}, \ldots, N_s)$

$$= \Sigma f(M_1, \ldots, M_r, N_1, \ldots, N_s),$$

where $M = M_1 \ldots M_l,\ N = N_1 \ldots N_k.$

Now f vanishes if $N_i > M_j$ $(i = 1, 2, \ldots, k,\ j = 1, \ldots, l)$
(i.e. if $N_1 > M_1, N_2 > M_1, \ldots, N_k > M_1$ or if $N > M_1^k$
also if $N_1 > M_2, N_2 > M_2, \ldots, N_k > M_2$ or if $N > M_2^k$

. . . .

. . . .

if $N_1 > M_l, N_2 > M_l, \ldots, N_k > M_l$ or if $N > M_l^k$
i.e. if $N_l > M^k$.

Thus f and therefore every term on the right side vanishes whenever $N^l > M^k$.

Hence ψ is a new type of ordinal function vanishing whenever $N^l > M^k$ and it is appropriate to refer to it as an indexed ordinal function.

NOTE. ψ vanishes whenever N^l does not divide M^k, for then there is a pair of values i, j, such that N_i does not divide M_j corresponding to every set of numbers $N_1 N_2 .. N_k = N$ and $M_1 ... M_l = M$ and therefore f vanishes by Theorem 1.

The following special cases are worthy of mention:—

(1) If $l = k$, then ψ reduces to an ordinal function of the first kind of type 1.

(2) If $l = r$, $k = s$, then ψ becomes an indexed ordinal function of two arguments.

(3) If f is a principal function, then considering N_i's first as minor arguments and then as major arguments, it follows by (2) that $\psi(M, N) = 0$ if $N^r >$ or $< M^s$.

ψ may be called a quasi-principal function which vanishes if $N^r \neq M^s$.

Lastly we shall establish the property of the convolute of the integral of an ordinal function of the second type l, k.

THEOREM 9. *The convolute of the integral with respect to the group of the k minor arguments of an ordinal function of the second type l, k is an indexed modular function.*

PROOF. Let Ω represent the convolution of $M_1, .., M_l$ and $N_1, .., N_k$ and let

$$F(M, M_{l+1}, .., M_r, N, N_{k+1}, .., N_s) = \Omega[f(M_1, .., M_r, N_1, .., N_s) \times E_0(M_1, .., N_{k+1}, N_s), E(N_1, .. N_k)]$$

$$= \Omega f(M_1, .., M_r, N_1, .., N_s)$$
$$\times \Omega E_0(M_1, .., N_{k+1}, .., N_s)\ E(N_1, .., N_k)$$
$$= \psi(M,\ M_{l+1}, .., M_r,\ N,\ N_{k+1}, .., N_s).$$
$$\times E_0(M,\ M_{l+1}, .., N_{k+1}, .., N_s)\ E(N)$$
$$= \sum \psi(d,\ d_{l+1}, .., d_r,\ \delta, .., \delta_s)\ E_0\left(\frac{M}{d}, .., \frac{N_s}{\delta_s}\right) E\left(\frac{N}{\delta}\right)$$
$$d/M,\ \delta/N,\ d_j/M_j\ (j = l+1, .., r)\ ;\ \delta_i/N_i\ (i = k+1, .., s),$$
$$= \sum \psi(M,\ M_{l+1} .. M_r,\ \delta, .. N_s)$$
$$= \sum_{\delta/N,\ \delta^l/M^k} \psi(M, .., M_{l+1}, .. M_r,\ \delta,\ N_{k+1},\ N_s).$$

Let $M = p_1^{m_1}\ p_2^{m_2} .. p_t^{m_t}$. Then $M^k = p_1^{m_1 k}\ p_2^{m_2 k}\ p_t^{m_t k}$. Therefore $M_{lk} = p_1^{\left[\frac{m_1 k}{l}\right]}\ p_2^{\left[\frac{m_2 k}{l}\right]} .. p_t^{\left[\frac{m_t k}{l}\right]}$, where M_{lk} is the greatest integer whose lth power divides M^k. If $g_{lk} = (M_{lk},\ N)$, it can be easily seen that if $\delta|N$, and $\delta^l|M^k$, then $\delta|g_{lk}$. Conversely any divisor δ of g_{lk} is such that $\delta|N$, and $\delta^l|M^k$. Hence $F(M,\ M_{l+1}, .., M_r,\ N, .., N_{k+1},\ N_s)$

$$= \sum_{\delta/g_{lk}} \psi(M,\ M_{l+1}, .., M_r,\ \delta,\ N_{k+1}, .., N_s).$$

Since $(M_{lk},\ N) = (M_{lk},\ N + \lambda M_{lk}) = g_{lk}$, where λ is any integer, it follows that F has the effective modulus M_{lk}.

Thus we have the concept of what may be called an *ndexed modular function.*

COR. (1) *The convolute of a modular function of the second kind of type* l, k (*viz. the result of convolving separately the group of modulus and the group of non-modulus arguments*) *is an indexed modular function.* (*Sec.* 3.1 (*b*) *and Th.* 5).

(2) *If* $l = r$, $k = s$, F *reduces to an indexed modular function of two arguments.*

STABILITY OF RECTILINEAR PLATES.

BY

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The stability of a rectangular plate has been the subject of a number of investigations. Bryan(1) has given the solution when the plate is supported at its edges. Southwell, Skan(2) and Dean(3) have also discussed the problem in a number of papers. When the edges of the plate are clamped, approximate solutions have been obtained by Sezawa,(4) Timoshenko,(5) Taylor,(6) Trefftz,(7) Weinstein,(8) Faxen,(9) Levy(10) and others. Recently Green(11) has obtained some of these results more directly by using the double Fourier series. We shall discuss the stability of rectilinear plates with supported edges under equal thrusts in two directions, and shall give some exact solutions for triangular plates.

Let a plane isotropic sheet be subjected to stresses P_1, P_2 parallel to rectangular axes, and let these stresses be such that they can maintain a small tranverse displacement w in the plate so contracted. The differential equation satisfied by w is known to be(12)

$$D\nabla^4 w + \left(P_1\frac{\partial^2}{\partial x^2} + P_2\frac{\partial^2}{\partial y^2}\right)w = 0, \tag{1}$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \text{ and } D = \tfrac{1}{12}\frac{Eh^2}{1-\sigma^2},$$

$2h$ being the thickness of the plate and E and σ having their usual meanings.

If we put $P_1 = P_2 = P$ (say), (1) can be written

$$\nabla^2(D\nabla^2 w + Pw) = 0. \tag{2}$$

For rectilinear plates supported at their edges the boundary condition can be written in the form

$$\nabla^2 w = 0,\ w = 0. \tag{3}$$

Thus if we can determine the solutions of

$$D\nabla^2 w + Pw = 0, \tag{4}$$

which satisfy the boundary condition $w=0$ we can use these very solutions for discussing the stability of supported plates. The problem of the stability of a rectilinear plate supported at its edges is therefore reduced to that of the transverse vibrations of a rectilinear membrane whose boundary is the same as that of the plate.

For the transverse vibrations of a rectangular membrane we have

$$w = W.\ \sin\frac{m\pi(x+a)}{2a}\sin\frac{n\pi(y+b)}{2b}, \tag{5}$$

the edges being $x=\pm a$, $y=\pm b$, m and n being integers. This gives

$$P = \tfrac{1}{4}D\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right), \tag{6}$$

which is what we get from Bryan's result.(1) For the stability of a square plate of side $2a$, P may not exceed $\frac{1}{2}D\pi^2/a^2$. It should be noted that this critical value of P corresponds to the gravest mode of vibration of the membrane.

For an isosceles right angled plate whose edges are $x=a, y=\pm x$, we have(13)

$$w = W\left[\cos\frac{(2m+1)\pi x}{2a}\cos\frac{(2n+1)\pi y}{2a} - \cos\frac{(2n+1)\pi x}{2a}\cos\frac{(2m+1)\pi y}{2a}\right], \tag{7.1}$$

$$P = \tfrac{1}{4}D\frac{\pi^2}{a^2}\left[(2m+1)^2+(2n+1)^2\right], \tag{7.2}$$

$$P_c = \tfrac{5}{2}\frac{D\pi^2}{a^2}, \tag{7.3}$$

P_c being the critical value of P.

For an equilateral plate with sides $x=a, y\sqrt{3}=\pm x$, we have(13)

$$v = W\left[\sin\frac{(m-n)\pi x}{a}\cos\frac{(m+n)\pi y\sqrt{3}}{a}\right.$$

$$\left.-\sin\frac{(2m+n)\pi x}{a}\cos\frac{n\pi y\sqrt{3}}{a}+\sin\frac{(2n+m)\pi x}{a}\cos\frac{m\pi y\sqrt{3}}{a}\right], \quad (8.1)$$

$$P = 4D\frac{\pi^2}{a^2}(m^2+mn+n^2), \quad (8.2)$$

$$P_c = 4D\frac{\pi^2}{a^2}. \quad (8.3)$$

For a right angled triangle containing an angle of 60° whose sides are $y = a/\sqrt{3}$, $x = a$, $y = x\sqrt{3}$, we get[13]

$$w = W\left[\sin\frac{(m-n)\pi x}{a}\sin\frac{(m+n+1)\pi y\sqrt{3}}{a}\right.$$

$$-\cos\left(2m+1+\frac{2n+1}{a}\right)\frac{\pi x}{a}\cos\frac{(2n+1)\pi y\sqrt{3}}{2a}$$

$$\left.+\cos\left(2n+1+\frac{2m+1}{2}\right)\frac{\pi x}{a}\cos\frac{(2m+1)\pi y\sqrt{3}}{2a}\right], \quad (9.1)$$

$$P = \frac{D\pi^2}{a^2}[(2m+1)^2+(2m+1)(2n+1)+(2n+1)^2], \quad (9.2)$$

$$P_c > \frac{D\pi^2}{a^2}. \quad (9.3)$$

Thus, if $(p/2\pi)$ be the frequency of the gravest mode of vibration of a rectilinear membrane, we have

$$\frac{P_c}{D} = \frac{p^2}{c^2}, \quad (10)$$

c being the ratio of the uniform tension and the superficial density ρ in the problem of the membrane.

Thus, even though the solution in (9) holds good for the symmetrical vibrations of an isosceles triangular membrane of vertical angle 120°, it cannot be used to determine P_c, as it does not give the frequency of the gravest mode. In such cases Rayleigh's method can be used to get the value of p.[14]

If A be the area of the plate we get the following comparison for the values of P_c.

Value of	Square.	Triangle. $\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi$	Triangle. $\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi$	Triangle. $\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{6}\pi$
$AP_c/D\pi^2$	2	2.31	2.50	2.69

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ON THE TRANSLATION NUMBERS OF INTEGRAL FUNCTIONS.*

BY

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1. Introduction. The object of this paper is to study the relation

$$f(z+\lambda)=g(z)f(z) \tag{1}$$

from two points of view, $f(z)$ and $g(z)$ being integral functions. In section I, I study the properties of integral functions $f(z)$ possessing one or more numbers λ satisfying an equation of form (1). In section II, I consider the solution of the functional equation (1) for a given λ and $g(z)$. It is known† that equation (1) always admits a meromorphic solution. But this is no longer true if we require that the solution of (1) should be an integral function. For instance, if $g(z)=\sin \pi z/\lambda$, equation (1) cannot have any solution in integral functions except the trivial solution $f(z)\equiv 0$. I consider also the conditions on $g(z)$ for the existence of a non-trivial solution in integral functions of the equation (1). The problem regarding the order of the solution is also discussed.

2. Definitions. Let $f(z)$ be any integral function. A number λ for which there exists an integral function $g(z)$ such that (1) holds, will be called a *translation number* of the function $f(z)$ provided $f(z)$ has *at least one zero*. This last restriction is to avoid the trivial case when for every number λ, the relation (1) holds. If λ and $-\lambda$ are both translation numbers of $f(z)$, we shall say that λ is a

* An abstract of this paper was communicated to the Fourteenth Conference of the Indian Mathematical Society held at Delhi in December 1945.

† J. M. Whittaker, *Interpolatory Function Theory* (Cambridge Tracts, No. 33), p. 31.

a *translation number of the first kind.* Otherwise λ will be called a *translation number of the second kind.* It is evident that if λ is a translation number, then so is $m\lambda$, where m is any *positive* integer. If λ is of the first kind, then $\pm m\lambda$ are also translation numbers. We shall prove below that there can be at most two independent translation numbers of the first kind while there can be an enumerable set of independent numbers of the second kind.

SECTION I.

3. TRANSLATION NUMBERS OF THE FIRST KIND. The main properties of translation numbers of the first kind are similar to those of periods of analytic functions. The following results, though elementary are proved for the sake of completeness.

THEOREM 1. *All translation numbers of the first kind on a straight line through $z = 0$ are of the form $\pm m\lambda$, where m is a positive integer.*

PROOF. The translation numbers of any integral function cannot have finite limit points. Hence on any straight line through the origin there is a translation number λ (if there be one at all) such that $|\lambda|$ is the lower bound of the moduli of translation numbers of $f(z)$ on this line. If μ is any translation number on this line, we can write $\mu = m\lambda + \lambda_1$, where m is an integer and $|\lambda_1| < |\lambda|$. Since λ is of the first kind, it follows that $\lambda_1 = \mu - m\lambda$ is also one translation number which is impossible unless $\lambda_1 = 0$. This proves the theorem.

3.1. THEOREM 2. *If $f(z)$ possesses two translation numbers of the first kind whose ratio is not real, then there exist two translation numbers λ_1 and λ_2 such that every translation number (whether of the first or second kind) is of the form $m_1\lambda_1 + m_2\lambda_2$, where m_1 and m_2 are integers. Moreover all translation numbers are of the first kind.*

PROOF. Since the translation numbers of an integral unction cannot have finite limit points, we can find a

translation number λ_1 having the least modulus and of the first kind. Among the remaining translation numbers of the first kind let λ_2 denote the one with the least modulus such that the vector $(0, \lambda_2)$ makes numerically the least angle (which is not zero, by hypothesis) with the vector $(0, \lambda_1)$. It is obviously enough to prove that any translation number λ whether of the first or second kind is of the form given in the theorem. We can write $\lambda = x_1\lambda_1 + x_2\lambda_2$, where x_1 and x_2 are real. If x_1 or x_2 or both be irrational we can find (by known properties of irrational numbers) an infinite number of triplets of integers (p, m_1, m_2) with $p > 0$ such that the set of numbers $\{ p\lambda + m_1\lambda_1 + m_2\lambda_2 = (px_1 + m_1)\lambda_1 + (px_2 + m_2)\lambda_2 \}$ has finite limit points, which is impossible since all these are translation numbers. If x_1 and x_2 be both rational, we can find a triplet (p_1, m_1, m_2) such that $p\lambda = m_1\lambda_1 + m_2\lambda_2$ so that $p\lambda$ and $-p\lambda$ are both translation numbers and hence $-p\lambda + (p-1\ \lambda = -\lambda$ is also a translation number. Hence λ is of the first kind. If x_1 or x_2 be an integer the result follows from the previous theorem. Let both be rational non-integers. Then we can find θ_1 and θ_2 such that $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and such that $\theta_1\lambda_1 + \theta_2\lambda_2$ is a translation number of the first kind. So $(1-\theta_1)\lambda_1 + (1-\theta_2)\lambda_2$ is also one of the same kind. Both these lie in the parallelogram whose adjacent sides are the vectors $(0, \lambda_1)$, $(0, \lambda_2)$. Also $|\theta_1\lambda_1 + \theta_2\lambda_2| < (\theta_1 + \theta_2)|\lambda_2|$ and $|(1-\theta_1)\lambda_1 + (1-\theta_2)\lambda_2| < (2-\theta_1-\theta_2)|\lambda_2|$ and one of these gives a contradiction in view of the choice of the numbers λ_1 and λ_2. Hence x_1 and x_2 are integers and our theorem is proved.

3.2. THEOREM 3. *A necessary and sufficient condition that λ should be a translation number of the first kind for $f(z)$ is that $g(z)$ in (1) should have no zeros. If $\{z_n\}$ denotes the zeros of $f(z)$, this statement is equivalent to the condition that the two sequences $\{z_n\}$ and $\{z_n + \lambda\}$ should be the same.*

PROOF. From (1), λ is a translation number of the first kind for $f(z)$ if and only if both $f(z+\lambda)/f(z)$ and

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$f(z-\lambda)/f(z)$ are integral functions. This is equivalent to saying that both $f(z-\lambda)/f(z)=g(z)$ and $f(z)/f(z+\lambda)=1/g(z)$ are integral functions which is possible if and only if $g(z)$ has no zeros. The second part of the theorem is obvious from the first part.

3·3. THEOREM 4. *(i) If λ is a translation number of the first kind for $f(z)$ then $f(z)=e^{h(z)}\,\theta(z)$, where $h(z)$ is an integral function and $\theta(z)$ is an integral function of period λ. The converse of this is also true.*

(ii) Let $f(z)$ have two translation numbers of the first kind whose ratio is not real. Let λ_1 and λ_2 (which will be called a pair of primitive translation numbers) be the two translation numbers of the first kind as described in Theorem 2. Let $\sigma(z)$ denote the canonical product associated with the lattice $\{m_1\lambda_1+m_2\lambda_2\}$, $m_1, m_2=0, \pm 1, \pm 2\ldots$ Then there exists a finite number of points $(z_1, z_2,\ldots, z_p)$ such that $f(z)=e^{h(z)}\prod_{i=1}^{p}\sigma(z-z_i)$. The converse is also true.

PROOF OF (*i*). Since the sequence $\{z_n\}$ of zeros of $f(z)$ is invariant for the transformation $z'=z\pm\lambda$, it follows* that there exists an integral function $\theta(z)$ with period λ and having the zeros $\{z_n\}$. Hence $f(z)$ is of the form given in the theorem. The converse is obvious.

PROOF OF (*ii*). Let $z_1, z_2, \ldots, z_p$ denote the incongruent zeros (counted with their proper multiplicity) of $f(z)$ in the parallelogram whose adjecent sides are the vectors $(0, \lambda_1)$ and $(0, \lambda_2)$. Then it is easily verified that $\prod_{i=1}^{p}\sigma(z-z_i)$ has precisely the same zeros as $f(z)$. Hence the result. Here also the converse is obvious.

3·4. REMARK. If $f(z)$ is of order ρ in Theorem 4 (*ii*), it follows that $\rho\geqslant 2$ and $h(z)$ is a polynomial of

* See my paper on 'Periodic Integral Functions' *J.I.M.S.*, (2) 5 (1941), 1-17.

degree not exceeding ρ. Since $\prod_{i=1}^{p} \sigma(z-z_i)$ is of order two, it follows that ρ must be an integer. The function $\sigma(z)$ is the usual Weierstrassian σ-function. Since it is known that $\sigma(z+\lambda) = e^{az+b}\,\sigma(z)$ (which can also be proved directly by using the oddness of $\sigma(z)$) it follows that $f(z+\lambda) = e^{P(z)} f(z)$, where $P(z)$ is of degree $\rho-1$ at most.

4. TRANSLATION NUMBERS OF THE SECOND KIND. The properties of translation numbers of the second kind are not so simple. The following theorem is obvious from definition.

THEOREM 5. *A necessary and sufficient condition that* λ *should be a translation number of the second kind for* $f(z)$ *is that the set* $\{z_n+\lambda\}$ *should be a proper subset of the sequence* $\{z_n\}$ *of zeros of* $f(z)$.

4.1. As already remarked, if λ is a translation number of the second kind for $f(z)$ then so are $m\lambda$, $m=1, 2, \ldots$ (but not $-m\lambda$). We shall now examine the number of independent translation numbers of the second kind which an integral function can possess. Obviously the set $T(f)$ of independant translation numbers of the second kind cannot possess finite limit points. So the set $T(f)$ is either finite or is a sequence tending to infinity. It will turn out that both can happen. The following theorem is obvious from Theorem 5.

THEOREM 6. (*i*) *In order that a sequence* $\{\lambda_n\}$ *should be translation numbers of the second kind for some integral function, it is necessary and sufficient that the set* $\{m_1\lambda_1+m_2\lambda+\ldots+m_p\lambda_p\}$, *where* $m_i \geqslant 0$ *are integers and* $p=1, 2, \ldots$ *should have infinity as its sole limit point.*

(*ii*) *In order that a finite set* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ *should be the translation numbers of the second kind for some integral function, it is necessary and sufficient that the set* $\{m_1\lambda_1+m_2\lambda_2+\ldots+$

$m_p\lambda_p \}$, $m_i \geqslant 0$ *being integers, should have infinity as its sole limit point.*

4.2. THEOREM 7. *Let λ be a translation number of the second kind. Then there cannot exist another translation number μ (whether of the first or second kind) such that λ/μ is real and negative.*

PROOF. If μ is of the first kind, then $-\mu$ is also a translation number. Hence it is enough to prove the theorem under the assumption that μ is of the second kind. Let $\lambda/\mu = -1/x$, $x > 0$. Then $\mu = -\lambda x$. Suppose x is irrational. Then $m\mu + n\lambda = (-mx+n)\lambda$ is also a translation number for all positive integral values of m and n. But it is known that the set $\{ -mx+n \}$ has every point as a limit point. So x cannot be irrational. Let $x = p/q$, where p and q are positive integers so that $\lambda/(-q) = \mu/(+p) = a\lambda + b\mu$, where a and b are positive integers such that $bp - aq = 1$. Hence $\lambda = -q(a\lambda + b\mu)$ is a translation number so that $-aq\lambda$ and so $-\lambda$ is also a translation number which is contrary to hypothesis. So the theorem is proved.

4.3. THEOREM 8. *Let λ_1 and λ_2 be two translation numbers of the second kind such that λ_1/λ_2 is not real. Then no number of the form $\lambda = -x_1\lambda_1 - x_2\lambda_2$, $x_1 \geqslant 0$, $x_2 \geqslant 0$ can be a translation number.*

PROOF. The cases $x_1 > 0$, $x_2 = 0$ and $x_1 = 0$, $x_2 > 0$ reduce to the previous theorem. So we shall suppose that both x_1 and $x_2 > 0$. If one or both of x_1 and x_2 be irrational then the set $\{ p\lambda + m_1\lambda_1 + m_2\lambda_2 \}$, p, m_1, m_2 being *positive* integers, will have finite limit points and this is impossible since each such number is a translation number. Let x_1 and x_2 be both rational. Then we can find positive integers p, m_1 and m_2 such that $p\lambda = -m_1\lambda_1 - m_2\lambda_2$. Hence $-m_2\lambda_2 = p\lambda + m_1\lambda_1$ is also a translation number so that $-\lambda_2 = -m_2\lambda_2 + (m_2 - 1)\lambda_2$ is also a translation number which is contrary to hypothesis.

4.4. REMARK 1. Similar arguments prove that if λ_1 is of the first kind and λ_2 of the second kind then no number of the form $x_1\lambda_1 - x_2\lambda_2$ with $x_2 > 0$ can be a translation number.

REMARK 2. It can now be seen that subject to the restrictions imposed by Theorems 7 and 8, the set $T(f)$ can be any finite set or set having its sole limit point at infinity. Using Theorem 6 it is easily seen that the following sets can be independent translation numbers of integral functions.

(i) $\{\sqrt{p_n}\}$, where $p_1 = 1, p_2, p_3, \ldots$ being the sequence of prime positive integers, (Theorem 7).

(ii) $1, i, \sqrt{2}(1+i), \sqrt{2}(1-i), -\sqrt{2}\,(1-i)$, (Theorem 8).

(iii) $1,\ i,\ \sqrt{p_n},\ \sqrt{p_n}+i,\ \sqrt{p_n}-i,\ n = 1, 2, \ldots$ where p_1, $p_2, \ldots$ are the primes, (Theorem 8).

SECTION II.

5. We shall now study the equation (1) as a functional equation. Firstly we have

THEOREM 9. *Let $g(z)$ in (1) be an integral function without zeros. Then (1) has always a solution $f(z)$. Any other solution of (1) will be of the form $f(z)\phi(z)$, where $\phi(z)$ is an integral function with period λ.*

PROOF. Let $g(z) = e^{h(z)}$. We always find* a function $P(z)$ such that $P(z+\lambda) - P(z) = h(z)$. The function $f(z) = e^{P(z)}$ is solution of (1). If $F(z)$ be any other solution then $F(z)/f(z)$ must have period λ and must be an integral function since $f(z)$ has no zeros. Hence $F(z) = \phi(z)\,f(z)$ as stated in the theorem.

5.1. REMARK. If $g(z)$ is of finite order ρ in Theorem 9, then ρ is an integer and $h(z)$ is a polynomial of degree ρ. Hence $P(z)$ can be chosen to be a polynomial of degree $\rho+1$. Hence $f(z)$ is of order $\rho+1$. Since the

* J. M. Whittaker, *l. c.* p. 25.

only periodic integral function of finite order without zeros is of order one and is of the form $Ae^{2\pi iz/\lambda}$ it follows that (1) cannot have a solution of order less than $1+\rho$ and the solution $f(z)$ without zeros and of finite order is unique except for a factor of the form $Ae^{2\pi iz/\lambda}$.

6. Next we consider the case when $g(z)$ in (1) has zeros. For the convenience of stating the results in a simple manner we shall suppose that $\lambda > 0$ in (1). We shall denote by $R(z)$ and $I(z)$ the real and imaginary parts of z.

THEOREM 10. *Let $\{z_n\}$ denote the zeros of $g(z)$ in (1). Then a necessary and sufficient condition that (1) should have an integral function solution $f(z)$ (other than $f(z) \equiv 0$) is that the set $\{z_n+m\lambda\}$, m, $n = 1, 2, \ldots$ should have no finite limit point.*

PROOF. The condition is necessary because if $f(z)$ is a solution of (1), the set mentioned in the theorem consists of the zeros of $f(z)$. Now suppose the condition is satisfied. Let $f(z)$ denote the canonical product with its zeros at $z_n+m\lambda$, m, $n = 1, 2, \ldots$. Then the zeros of $f(z+\lambda)$ will be $\{z_n+(m-1)\lambda\}$, m, $n = 1, 2, \ldots$. Hence $f(z+\lambda)/f(z)$ is an integral function having the same zeros $\{z_n\}$ as $g(z)$. Here $f(z+\lambda) = e^{h(z)}\, g(z)\, f(z)$. Choose $P(z)$ so that $P(z+\lambda)-P(z) = -h(z)$. Then $F(z) = e^{P(z)} \times f(z)$ is a solution of (1).

6.1. In the statement of the following theorem we agree that if a sequence $\{z_n\}$ contains a particular number α repeated *infinitely often*, then α *is a limit point* of the sequence.

THEOREM 11. *Let $\{z_n\}$ denote the zeros of $g(z)$ in (1) where $\lambda > 0$. Then a necessary and sufficient condition that (1) should have a non-trivial integral function solution is that the set $\{I(z_n)\}$ for which $R(z_n) < 0$ should have no finite limit point.*

PROOF. THE CONDITION IS SUFFICIENT. By Theorem 10, it is enough to prove that the number of points of the sequence $E = \{z_n + m\lambda\}$, $m, n = 1, 2, \ldots$ in any circle C_r: $|z| \leqslant r$, is finite. [It is to be noted that any point $z = z_m + m\lambda$ of E is to be counted as many times as it could be obtained starting from the zeros of $g(z)$ and adding positive integral multiples of λ.] Let $z = x + iy$. Now the number of $\{z_n\}$ in each of the rectangles $p\lambda \leqslant x \leqslant (p+1)\lambda, |y| \leqslant r, p = 0, 1, \ldots$ is finite. Hence the number of points of E in C_r arising from those $\{z_n\}$ for which $R(z_n) \geqslant 0$ is finite. By hypothesis we can find p so great that there are no points of $\{z_n\}$ with $R(z_n) < -p\lambda$ and $|I(z_n)| \leqslant r$. Hence the points of E in C_r arising from those $\{z_n\}$ with $R(z_n) < 0$ is also finite. Hence the number of points of E in C_r is finite.

THE CONDITION IS NECESSARY. By Theorem 10, the set E cannot have finite limit points and E will be the zeros of the solution $f(z) \not\equiv 0$. Suppose now that the condition of the theorem is not satisfied. We shall show that this leads to a contradiction. We consider three cases:—

CASE (I). The $\{I(z_\mu)\}$ with $R(z_n) < 0$ contains an infinity of distinct points having a limit point α. Then the rectangle $0 \leqslant x \leqslant \lambda$, $|y - \alpha| \leqslant 1$ will contain an infinity of distinct points of E which is impossible.

CASE (II). $I(z_n) = \theta$ for all but a finite number of values of n for which $R(z_n) < 0$. We may suppose $\theta = 0$ so that $z_n = R(z_n) < 0$. If the set $\{z_n < 0\}$ contains a sub-sequence such that no two terms of this sub-sequence differ by multiples of λ, then the segment $0 \leqslant x \leqslant \lambda, y = 0$ contains an infinity of distinct points of E which is impossible.

CASE (III). Suppose now that under case (ii) $(\theta = 0)$ the zeros $z_n < 0$ are such that all but a finite number of

them differ from one another by multiples of λ. If θ_1 and θ_2 are zeros of $g(z)$ of orders p and q and $\theta_1 = \theta_2 + m\lambda$ for some positive integral m, then $\theta_1+\lambda$ is a zero of $f(z)$ of order at least $p+q$ as is seen from (1). Using this result we see that $z_i+\lambda$ (which z_i is the first zero of $g(z)$ with $z_n < 0$) is zero of infinite order for $f(z)$ in the case under consideration, that is, E contains one particular number repeated infinitely often. This also is impossible. Hence the condition is necessary. So the theorem is proved.

6.2. Remark. It follows from Theorem 11, that the solution $f(z)$ of (1) envisaged in the proof of Theorem 10 cannot have zeros of the form $z_0 - m\lambda$ for all $m = 1, 2, \ldots$ since these will give rise to zeros of $g(z)$ of the form $z_i - m\lambda$ for an infinity of positive integral values of m, where z_i is a zero of $g(z)$; this is not possible by Theorem 11. This solution $f(z)$ of (1) when $g(z)$ is a function with zeros will be called a *canonical solution* and is unique except for a factor without zeros.

6.3. Theorem 12. *Let $g(z)$ have zeros and let $f(z)$ be a canonical solution of* (1). *Then any other solution of* (1) *is of the form $f(z)\phi(z)$, where $\phi(z)$ is an integral function with period λ.*

Proof. Let $F(z)$ be any solution of (1) and $\phi(z) = F(z)/f(z)$. Then $\phi(z)$ is a meromorphic function with period λ. Since $F(z) = f(z)\,\phi(z)$ is an integral function, it follows that the poles, if any, of $\phi(z)$ must be zeros of $f(z)$. If z_0 is a pole of $\phi(z)$ then $z_0 \pm m\lambda$, $m = 1, 2, 3, \ldots$ must also be poles. Hence $z_0 - m\lambda$ must be zeros of $f(z)$ for all $m = 1, 2, \ldots$ which is impossible as explained in §6.2. Hence $\phi(z)$ cannot have any poles so that it is an integral function.

6.4. As explained in §5.1, if $g(z)$ in (1) has no zeros and is of order ρ, there is a solution $f(z)$ of order $\rho+1$ for (1). But this is not in general true when $g(z)$ has zeros. For instance if $g(z)$ is the canonical product

with the zeros $\{-n+i\log n\}$, $n=1, 2,\ldots$, then $g(z)$ is of order one. But the zeros of the canonical solution (when $\lambda=1$) will be $\{m-n+i\log n\}$, $m, n=1, 2,\ldots$. But $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 1/|m-n+i\log n|^{\rho}$ does not converge for any finite ρ since

$$\sum_{n=1}^{\infty}\frac{1}{|m-n+i\log n|^{\rho}}\geqslant\frac{1}{(\log m)^{\rho}},\ m\geqslant 2.$$

But with an additional restriction on the situation of the zeros of $g(z)$ we can prove that (1) has a solution of order $\rho+1$ if $g(z)$ is of order ρ.

THEOREM 13. *Let* $\{z_n\}$, $z_n=r_n e^{i\theta_n}$, *be the zeros of* $g(z)$ *which is of order* ρ. *Let* $\delta>0$ *exist such that* $|\theta_n|\leqslant\pi-\delta$. *Then* (1) *has a solution of order* $1+\rho$.

PROOF. Under the conditions of the theorem, $\{z_n\}$ satisfies the conditions of Theorem 11. In view of the results in §§5 and 5.1, it is enough to prove that if the exponent of convergence of $\{z_n\}$ be ρ, then that of $\{z_n+m\lambda\}$, $m=1, 2,\ldots$ is $\rho+1$. Now let $\mu>1$. Then

$$0\geqslant\sum_{m=1}^{\infty}\frac{1}{|z_n+m\lambda|^{\mu}}-\int_1^{\infty}\frac{dt}{|z_n+\lambda t|^{\mu}}\leqslant\frac{1}{|z_n+\lambda|^{\mu}}$$

by the usual integral test. Putting $\lambda t=r_n v$ (we suppose $r_n\neq 0$) we get

$$\int_1^{\infty}\frac{dt}{|z_n+\lambda t|^{\mu}}=\frac{1}{\lambda r_n^{\mu-1}}\int_{\lambda/r_n}^{\infty}\frac{dv}{|v+e^{i\theta_n}|^{\mu}}.$$

Since $|\theta_n|\leqslant\pi-\delta$, it follows that $\sqrt{(v^2+1-2v\cos\delta)}\leqslant|v+e^{i\theta_n}|\leqslant v+1$. This shows that there exist fixed positive numbers A and B such that

$$\frac{A}{r_n^{\mu-1}}\leqslant\sum_{m=1}^{\infty}\frac{1}{|z_n+m\lambda|^{\mu}}\leqslant\frac{B}{r_n^{\mu-1}}$$

for all n after a certain stage. Hence the exponent of

convergence of $\{z_n+m\lambda\}$ is $\rho+1$ if that of $\{z_n\}$ be ρ. As the example cited above shows, if the condition on $\{z_n\}$ given in the theorem be *violated*, there *may be no solution of finite order* at all for (1).

6.5. Some special cases.

(1) $g(z)=a\neq 0$. Here $f(z)=e^{z/\lambda . \log a}$ is a solution of (1) $(\log \lambda+\pi i)\ z/\lambda$.

(2) $g(z)=z$. Here $f(z, 0)=e^{z/\lambda . (\log \lambda+\pi i)}$ is a canonical solution of (1).

(3) $g(z)=1-\dfrac{z}{a}$. Here

$$f(z)=f(z, a)=e^{z/\lambda\ (\log \lambda/a)}\cdot \frac{1}{\Gamma\left(\frac{a-z}{\lambda}\right)}$$

is a canonical solution of (1).

(4) $g(z)=a\, z^h \prod\limits_{i=1}^{n}\left(1-\dfrac{z}{a_i}\right)$. Here

$$f(z)=e^{z/\lambda . \log a}\{f(z, 0)\}^h \prod_{i=1}^{n} f(z, a_i)$$

is a canonical solution.

(5) As an illustration of Theorems 11 and 13, let $\lambda=1$ and $g(z)=\dfrac{1}{\Gamma(-z)}$. Let $\phi(z)=\prod\limits_{n=1}^{\infty}\left(1-\dfrac{z}{n}\right)^n e^{z+z^2/2n}$. Then we can find a and b so that

$$f(z)=e^{az^2+bz}\phi(z)$$

is a canonical solution of (1) and this is the only canonical solution of finite order (here $\rho=1$ and $\rho+1=2$) but for a constant factor. A little calculation shows that $a=\frac{1}{2}(\gamma-1)$, $b=-\frac{1}{2}(\gamma-1)-i\pi-\log\phi'(1)$, where γ is Euler's constant. On the other hand if $g(z)=\dfrac{1}{\Gamma(z)}$, the equation (1) has no other solution than $f(z)\equiv 0$.

THE RING OF ENDOMORPHISMS FOR WHICH EVERY SUBGROUP OF AN ABELIAN GROUP IS INVARIANT.

BY

F. W. LEVI, (*Calcutta*).

[Received 14 October, 1946].

Given an additive Abelian group A, the invariance of a subgroup B of A for an endomorphism α of A is a dyadic relation $\alpha\rho B$ between endomorphisms and subgroups of A. From every dyadic relation, two reciprocal lattices of "closed" sets are derived*. In the present case, the closed sets of endomorphisms are rings and the closed sets of subgroups are all the possible systems of admissible subgroups of A. To the largest group-theory which A affords, there corresponds the ring of those endomorphisms for which every subgroup of A is invariant. This ring $R(A)$ contains obviously all the endomorphisms $\{m\}$ which map $a \to ma$, but it may also contain other endomorphisms. It will be shown that $R(A)$ depends only on the orders of the elements which occur in A. If there is in A an element of order 0, then $R(A)$ is isomorphic to the ring of the integral numbers; otherwise it is homomorphic to the direct sum of rings of p-adic numbers, p running over all the prime numbers.

Obviously an endomorphism α belongs to $R(A)$ if and only if every cyclic subgroup remains invariant for α. Hence α maps $a \to ma$, where m is determined mod *order* a only, i.e. to every a, a class of residues $M(a)$ mod *order* a is coordinated such that α maps a on ma if and only if $m \in M(a)$. If *order* $a = 0$ (i.e. if a generates an infinite cyclic group), then $M(a)$ consists of a single number. The

* Garrett Birkhoff: 'Lattice theory', *Amer. Math. Soc. Coll.* XXV, p. 24-26.

lowest common multiple of the orders of the elements of a group G is denoted by *exp* G; thus *exp* $G = 0$ when there exists in G an element of order 0, or when there is no upper bound for the orders. Let a, b be two arbitrary elements of the given Abelian group A, and B the subgroup generated by them, then B admits a basis c, d and $\alpha \in R(A)$ maps $c \to kc$, $d \to md$, $c+d \to n(c+d)$. Hence $nc = kc$, $nd = md$, and therefore B is mapped by α in the same way as by $\{n\}$. The corresponding holds for every subgroup generated by a finite number of elements:

$M(a_1)$, $M(a_2)$, ..., $M(a_r)$ *have a common element.*

Hence if *order* $(a_1) = 0$, then α is equivalent with some $\{n\}$ and $R(A)$ is isomorphic to the ring of the integral numbers. When A is infinite, $R(A)$ is homomorphic to the ring of the endomorphisms $\{n\}$ and isomorphic to the ring of the classes of residues mod *exp* A. Further *order* a = *order* b implies $M(a) = M(b)$ and when *order* a is a factor of *order* b, then $M(a) \supseteq M(b)$.

Suppose A is infinite and every element is of order $\neq 0$, then A is the direct sum of its primary components

$$A = P_1 + P_2 + \dots$$

Consider one primary component P belonging to the prime number p. If *order* $a = p^r$, *order* $b = p^{r+s}$, then the elements of $M(a)$ are congruent $c_r = d_0 + d_1 p + \dots + d_{r-1} p^{r-1} \pmod{p^r}$ and those of $M(b)$ are congruent $c_r + d_r p^r + \dots + d_{r+s-1} p^{r+s-1} \pmod{p^{r+s}}$. Hence we obtain all the sets M from initial sections of one p-adic number. On the other hand, let such a p-adic number be given and c_r its initial sections for $r = 1, 2, \dots$, then the mapping α: $a \to c_r a$ (for p^r = *order* a) leaves every cyclic group invariant and as $c_r a = c_{r+s} a$, $\alpha(a+b) = \alpha a + \alpha b$. Hence α is an endomorphism and therefore $\alpha \in R(P)$. This ring is therefore homomorphic to the ring of the p-adic numbers. If *exp* $P = 0$, the homomorphism is an isomorphism; if

exp $P = p^m$, then $R(P)$ is isomorphic to the ring of the classes of residues mod p^m.

Let A be a direct sum $A = S + T$, where the orders of the elements of S are relatively prime to the orders of the elements of T and therefore there is no element of order 0 in A. Then every $\alpha \in R(A)$ generates in S a $\beta \in R(S)$ and in T a $\gamma \in R(T)$. We shall show that conversely when β and γ are given, the mapping α:

$$\alpha(s+t) = \beta s + \gamma t \text{ (when } s \in S, t \in T)$$

belongs to $R(A)$ and therefore $R(A)$ is a direct sum

$$R(A) = R(S) + R(T).$$

Given two arbitrary elements of A: $a' = s' + t'$ and $a'' = s'' + t''$, then s', s'' generate a finite subgroup S_1 of S and similarly t', t'' generate $T_1 \subseteq T$. *Exp* $S_1 = \sigma$, *exp* $S_2 = \tau$, $(\sigma, \tau) = 1$. In S_1, β is equivalent to some $\{m\}$ and in T_1, γ is equivalent to $\{n\}$ both groups being finite. Then there exists a number $w \equiv m \pmod{\sigma} \equiv n \pmod{\tau}$ and α is equivalent to $\{w\}$ in $S_1 + T_1$. Hence $\alpha(a' + a'')$ and the cyclic group generated by a' is invariant. Thus $\alpha \in R(A) = R(S) + R(T)$. By mathematical induction we obtain

$$R(A) = R(P_1) + R(P_2) + \dots.$$

To obtain $R(A)$, we have therefore to know only the lattice of the orders of the elements of A (the lattice operations being h.c.f. and l.c.m.). If the lattice has a supremum, e.g. zero, then it is $=$ *exp* A and $R(A)$ is isomorphic to the ring of the classes of residues mod *exp* A. If there is no supremum, one has to consider the sublattices of the prime number powers that occur. If the sublattice belonging to the prime number p is of finite dimension n, then $R(P)$ is isomorphic to the ring of the residue classes mod p^n, otherwise to the ring of the p-adic numbers.

A NOTE ON THE PARITY OF $p(n)$

BY

HANSRAJ GUPTA.

[Received 26 August, 1946].

In 1920, Mac Mahon* gave a method for determining the parity of $p(n)$—the number of unrestricted partitions of n. He made use of Jacobi's Elliptic products. His results are here obtained in a simpler and more direct way.

We have

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\left\{ \prod_{r=1}^{\infty} (1-x^r) \right\}^{24},$$

so that

$$\left\{ \prod_{r=1}^{\infty} (1-x^r)^{32} \right\}^{-1} \cdot \sum_{n=1}^{\infty} \tau(n)\, x^{n-1} = \left\{ \prod_{r=1}^{\infty} (1-x^r)^{8} \right\}^{-1}.$$

Since

$$(1-x^r)^8 \equiv 1-x^{8r} \pmod 2, \tag{1}$$

$$(1-x^r)^{32} \equiv 1-x^{32r} \pmod 2, \tag{2}$$

and†

$$\tau(n) \text{ is odd if and only if } n \text{ is an odd square}, \tag{3}$$

we get

$$\left\{ \prod_{r=1}^{\infty} (1-x^{32r}) \right\}^{-1} \cdot \sum_{i=0}^{\infty} x^{(2i+1)^2-1} \equiv \left\{ \prod_{r=1}^{\infty} (1-x^{8r}) \right\}^{-1} \pmod 2.$$

i.e.

$$\sum_{m=0}^{\infty} p(m)\, x^{32m} \cdot \sum_{i=0}^{\infty} x^{4i(i+1)} \equiv \sum_{n=0}^{\infty} p(n)\, x^{8n} \pmod 2.$$

Comparing the coefficients of like powers of x, we have

$$p(n) \equiv \sum_t p(t) \pmod 2, \tag{4}$$

* MacMahon, *Proc. Cambridge Phil. Soc.* 20 (1921), 281-3.

† Hansraj Gupta, *Proc. Benares Math. Soc.* 5 (1943) 17-22.

where t runs through the positive integral values given by the relation

$$t=\frac{2n-i(i+1)}{8},\ i\geqslant 0.$$

Evidently, we need give to i only those values for which

$$i(i+1)=2n \pmod 8. \qquad (5)$$

These results are in agreement with those of MacMahon.

ON THE THE ABERRANCY CURVE.*

BY

B. K. SEN GUPTA, *Srikhanda, (Bengal).*

4. The origin is an undulation on the curve $y=px+qx^2/2!+\ldots$ if $q=r=0$. The point on the aberrancy curve corresponding to the origin is $x=-3qr/P$, $y=3q(3q^2-pr)/P$, where $P=3qs-5r^2$. When $q=r=0$, Lt. $dP/d\theta=3qt-7rs=0$ and Lt. $d^2P/d\theta^2=-7s^2$. The limiting values of x and y at the origin are respectively given by $-3r$. Lt. $q/P=-3r$. Lt. $r/dp/d\theta=-3r$. Lt. $s/d^2p/d\theta^2=3r/7s=0$, and $3(3q^2-pr)$. Lt. $q/P=0$, i.e. $x=0$, $y=0$ on the aberrancy curve. So the origin also lies on the aberrancy curve.

For the aberrancy curve dy/dx at the origin is $(pr-3q^2)/r=p-3q^2/r=p$ since Lt. $q^2/r=0$. Hence dy/dx is the same as that of the given curve; so at the point of undulation, the curve and the aberrancy curve have got the same tangent.

d^2y/dx^2 of the aberrancy curve is q. P^3/r^3. $T=q$. (r^3/T). $(P/r^2)^3$, where T (the Mongian) $=40r^3-45qrs+9q^2t$; Lt. $dT/d\theta=0$ at the origin; Lt. $T/r^3=35/2$ or Lt. $r^3/T=2/35$; Lt. $P/r^2=-7/2$; hence $d^2y/dx^2=q.\lambda$ (constant) $=0$.

*This section was inadvertently omitted from the paper, "On the Aberrancy curve", J.I.M.S. Vol. IX, 77-9. Ed.

d^3y/dx^3 of the aberrancy curve $= P^4. \ [P(r^2T-3qsT -qr.\,dT/d\theta)+3qrT.\,dp/d\theta]/r^5.\ T^3 = (P/r^2)^5.\ (r^3/T)^2(r-3qs/r -q/T.\ dT/d\theta)+3(P/r^2)^4.\ (r^3/T)^2.\ q/r^2.\,dp/d\theta.$ Now Lt. $3qs/r = 3s$. Lt $q/r = 0$; Lt. $q/T.\ dT/d\theta = dT/d\theta$. Lt. $q/T = dT/d\theta.\ r/dT/d\theta = 0$; Lt. $q/r^2.\ dp/d\theta = dp/d\theta$. Lt. $q/r^2 = dp/d\theta.\ 1/2s = 0$. Hence $d^3y/dx^3 = 0$.

From the above statements we come to the following

THEOREM. *The undulations on a given curve are also undulations on the aberrancy curve, and the tangents at them are the same for both.*

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NEXT PRIZE PROBLEM.

To make a contribution to the theory of plane projective geometries particularly the following types of non-desarguesian geometries:

(*i*) The geometries where a single (p, L) Desargues theorem holds, where L is a line passing through the point p. [The (p, L) Desargues theorem is that which states that if the three joins of the corresponding vertices of two triangles concur at p, and if two pairs of their corresponding sides intersect on L, then the third pair also intersect on L.]

(*ii*) The geometries in which there is a transitive group of translations.

(*iii*) The geometries in which the theorem of the complete quadrangle (that is, the existence of a unique harmonic conjugate C' of C with respect to AB for every triad ABC of collinear points) holds.

(*iv*) The required "contribution" is to bear on the following questions:—

1. The distribution of points p and lines L passing through p, such that the (p, L) Desargues theorem is true, the distribution of collinear triads of points A, B, C, such that C has a unique harmonic conjugate with respect to AB.

2. The distribution of quadrangles with three collinear diagonal points.

3. The projective group of the plane and the projective groups of its straight lines.

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